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OPTIMAL OPERATION OF A TIDAL TURBINE

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Abstract: The optimum operation strategy of a tidal turbine is calculated for a generic problem which consists of a small and shallow channel between two infinite reservoirs exposed to tidal waves. The optimal operating volume flow rate is found to be $\mathbf{1/\sqrt{3}}$ of the natural flow rate without employed tidal turbines and yields the maximum energy to be harvested per period, which is $\mathbf{2\sqrt{3}/9 \approx 38.5\%}$ of the initially transported energy per period.

1 Introduction

Presumably, turbine arrays will be installed in tidal channels to harvest tidal power in the future at several sites on earth. One example for a well suited site is the Inner Sound between the Island of Stroma and the shore of Northern Scotland (fig 1).

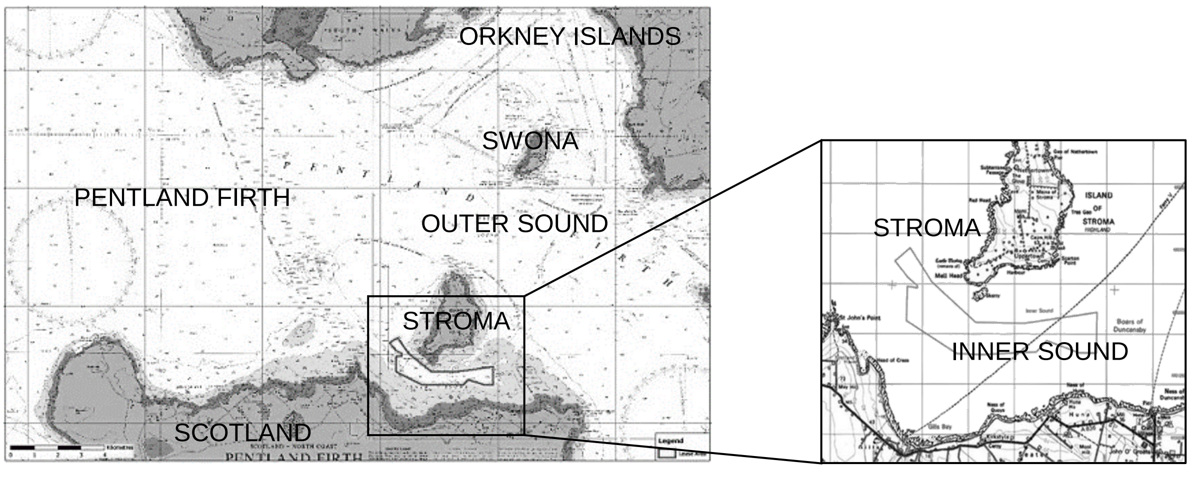

Fig. 1: Tidal channel between the island of Stroma and the shore of Scotland, situated in the tidal system of the Pentland Firth.

It is important to notice that the volume flow rate in a tidal channel is not a given quantity but the result of the height difference applied to the surrounding system. However, as the open ocean is an infinite reservoir compared to the capacity of the tidal system one can take its water head as a given quantity and hence as boundary conditions for the dynamic system of the tidal channel.

Figure 2 shows a generic model of a tidal channel in an abstract manner employing lumped parameters. The tidal channel is connecting two basins I and II, and is situated within a greater tidal system including a possible bypass flow like the outer sound in the tidal system of the Pentland Firth.

As discussed by Pelz & Metzler [1] the turbine array can herein be treated as a nonlinear resistance showing negligible inductance and capacity. In other words: the flow through the turbine array can be treated quasi-stationary. This is true as long as the time $l/\sqrt{gh}$ needed for a wave to travel through the channel (which is $< 10^3$s for a channel as long as $10\ \mathrm{km}$ and as shallow as $10\ \mathrm{m}$) is much smaller than the tides' cycle time $T \approx 4.5\times10^4\ \mathrm{s}$.

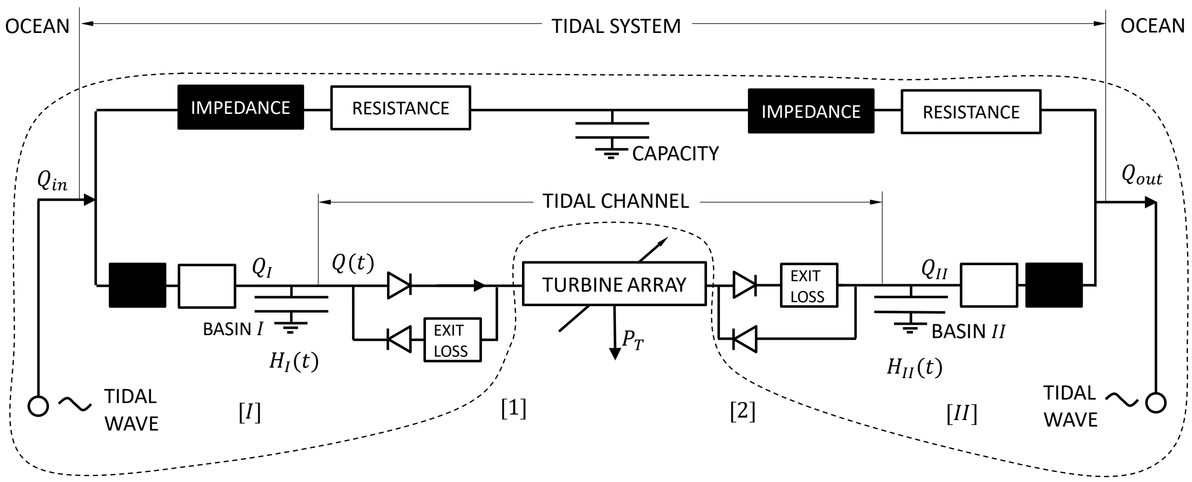

Fig. 2: Lumped parameter model of a tidal channel.

If the tidal waves at either side of the tidal system differ, the applied height difference results in a certain state of the system. By operating tidal turbines in the tidal channel the flow condition in the system is affected.

When operating the turbine field, the operator aims to maximize the mechanical work gained from one tidal circle:

$$\max W_T = \max \int_0^T P_T \,\mathrm{d}t\,,$$

which may be written as an variational problem [2]

$$\delta W_T = \delta \int_0^T P_T \,\mathrm{d}t = 0.$$

In this contribution the variational problem is solved for a generic case in order to answer two principle questions: What is the maximal work to be harvested? And secondly, how is the turbine field to be operated to gain this maximal work in one tide cycle?

2 Formulation of the generic optimization problem

Modelling a tidal system, one would describe the system within the dashed line in figure 2 by a shallow water model, which requires the detailed topology of the site. In other words, the impedances, resistances and capacities are site specific.

This problem specific consideration is not in the focus of the presented research. In contrast, a simple generic relaxation problem (fig. 3) is considered in order to obtain general results.

The generic channel of width b connects two infinite reservoirs I and II which are exposed to phase-shifted tidal waves with energy heights $H_I(t)$ and $H_{II}(t)$. The bed of the shallow tidal channel has the elevation z_1 and z_2 above reference (see fig. 3). In the context of the example the channel reflects the inner sound connecting the two sides of the Pentland Firth which are open to the relatively infinite reservoirs of the North Atlantic and the North Sea respectively. Any effects of hydraulic impedances and capacities are neglected as is the effect of the outer sound which is justified as long as $\sqrt{l/g} \ll T$ [1].

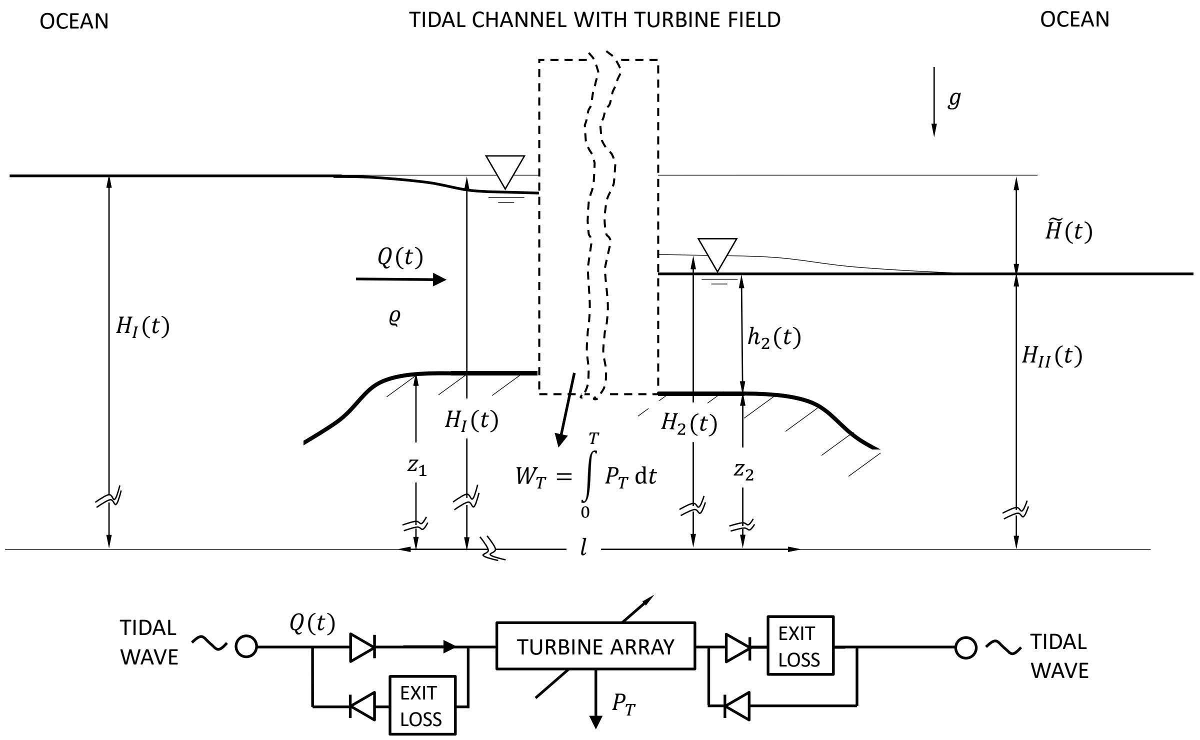

Fig. 3: Generic channel composed of a basin I relaxing through a turbine array in an infinite basin II.

The tidal channel is assumed to have a constant width b. Hence, it is convenient to introduce the specific volume flow $q = Q/b$.The energy equation per unit depth for the sketched control volume (cf. dashed line) in figure 3 then reads

$$P_T(t) = \eta\, \varrho g b\, q(t)\, [H_1(t) - H_2(t)]. \quad (1)$$

Here ϱ denotes the water density and g the specific gravity constant. In (1) the efficiency η is a dimensionless measure for all dissipative losses within the turbine field. This includes the losses within the turbines itself but also the mixing losses downstream of each turbine (see Pelz et. al. [1]). But as we are considering a generic system we assume η to be constant, i.e. we neglect the dependence of the mixing losses on the volume flow q in this paper.

For unsymmetrical systems the exit losses differ when changing the flow direction. However, the derivation of the optimal operation strategy remains. For the direction of flow as sketched in figure 3 the water heads up- and downstream of the turbine are $H_1(t) = H_I(t)$ and $H_2 = H_{II} + q^2/(2gh_2^2)$, respectively. The last term $q^2/(2gh_2^2)$ accounts for the exit loss (i.e. Carnot loss) due to mixing of the channel flow with the standing water in reservoir II. $h_2 = H_{II} - z_2$ is the downstream water level in the tidal channel. (For the unlike case of supercritical downstream flow a hydraulic jump may appear. This special and for practical applications unimportant case is not considered here.)

With the abbreviations

$$\widetilde{H}(t) := H_I(t) - H_{II}(t) \qquad (2.1)$$

$$R(t) := \begin{cases} \dfrac{1}{2g\,[H_{II}(t) - z_2]^2}, \text{for } H_I \geq H_{II} \\ \dfrac{1}{2g\,[H_I(t) - z_1]^2}, \text{for } H_I < H_{II} \end{cases} \qquad (2.2)$$

the turbine shaft power writes

$$P_T(t) = \varrho g b\, \eta\, q(t)\left[\widetilde{H}(t) - R(t) q(t)^2\right]. \qquad (3)$$

3 Solving the variational problem

The fundamental lemma of calculus of variations says that if a function $x(t)$ minimizes or maximizes the functional

$$J(x) := \int_{t_0}^{t_1} f[t, x(t), \dot{x}(t)]\mathrm{d}t$$

(considering all in $[t_0, t_1]$ continuously differentiable equations satisfying the boundary conditions $x(t_0) = x_0$ and $x(t_1) = x_1$), then $x(t)$ is a solution of the Euler-Lagrange equation [2]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) = \frac{\partial f}{\partial x}.$$

Hence, the optimal volume flow rate $q(t)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial P_T}{\partial \dot{q}}\right) = \frac{\partial P_T}{\partial q}. \qquad (4)$$

As long as impedance is small which is indeed the case for $\sqrt{l/g} \ll T$, the left hand side of the Euler-Lagrange equation vanishes. The variational problem in this case reduces to $\partial P_T / \partial q = 0$ or

$$\widetilde{H}(t) - 3R(t)q(t)^2 = 0. \quad (5)$$

Thus, the optimal flow rate (in other words the flow rate which results if the turbine field is optimally operated when considering the whole system) is

$$q_{opt}(t) = \sqrt{\frac{1}{3}\frac{\widetilde{H}(t)}{R(t)}}. \quad (6)$$

Inserting equation (6) into (3) yields the corresponding turbine power

$$P_{T,opt}(t) = \eta\, \frac{2\sqrt{3}}{9} \varrho g b \sqrt{\frac{\widetilde{H}(t)^3}{R(t)}}\,, \quad (7)$$

which yields the maximal work that can be gained from one tidal cycle

$$W_{T,max} = \eta \frac{2\sqrt{3}}{9}\, \varrho g b \int_0^T \sqrt{\frac{\widetilde{H}(t)^3}{R(t)}}\; \mathrm{d}t. \quad (8)$$

4 Formulation of a general principle

Optimal flow rate $q_{opt}(t)$, turbine power $P_{T,opt}(t)$ and maximal work $W_{T,max}$ are functions of the available tidal height difference at the site $\widetilde{H} \coloneqq H_I - H_{II}$ and the loss-coefficient $R(t)$ which itself depends on the exit sides water level and ground elevation. In order to obtain a more general principle, the unaffected channel flow $q_0(t)$ is used as a reference.

When no tidal turbine is employed, the flow rate through the channel ensues such that the losses in the channel balance the height difference applied which yields

$$\widetilde{H}(t) = R(t)\, q_0^2(t) \quad (9)$$

and hence

$$q_0(t) = \sqrt{\frac{\widetilde{H}(t)}{R(t)}}. \quad (10)$$

Combining equations (6) and (10), a simple, but general principle is derived:

> *The tidal turbines in the generic channel (with constant efficiency) are operated at best if the power extraction results in a reduction of the channel flow to* $1/\sqrt{3}$ *of the unaffected flow:*

$$q_{opt}(t) = \frac{1}{\sqrt{3}}\, q_0(t) \qquad (11)$$

Inserting equation (10) into equation (7) the power writes

$$P_{T,opt}(t) = \varrho g b\, \eta \frac{2\sqrt{3}}{9} \widetilde{H}(t) q_0(t) = \eta \frac{2\sqrt{3}}{9} P_0(t), \quad (12)$$

with $P_0(t) :=\ \varrho g b\, \widetilde{H}(t) q_0(t)$ being the power initially transported through the channel before employing and operating tidal turbines within the channel.

The available work per cycle is

$$W_{avail} \coloneqq \int_0^T P_0(t)\, \mathrm{d}t =\ \varrho g b \int_0^T \widetilde{H}(t) q_0(t)\, \mathrm{d}t\ .$$

Hence the cycle efficiency $C_W \coloneqq W_T / W_{avail}$ is limited by

$$C_W \leq \frac{W_{T,max}}{W_{avail}} = \frac{2\sqrt{3}}{9} \eta = 0.385 * \eta.$$

5 Conclusion

The volume flow rate in a tidal channel is not a given quantity but the result of the height difference applied to the surrounding system. By operating tidal turbines in the tidal channel the flow condition in the system is affected, speaking the volume flow rate is reduced with increasing turbine thrust. When operating the turbine field, the operator hence aims to maximize the mechanical work gained in one tidal circle which leads to the variational problem $\delta\, W_T = \delta \int_0^T P_T\, \mathrm{d}t = 0$.

In this paper the problem is solved for a generic channel with neglectable hydraulic impedance and capacity connecting two infinite reservoirs which are exposed to different tidal waves $H_I(t)$ and $H_{II}(t)$.

The characteristic properties of the generic system are the applied height difference $\widetilde{H}(t) = H_I(t) - H_{II}(t)$ and the exit loss (i.e. Carnot loss) $R(t) q^2(t)$. They can be condensed in the unaffected volume flow $q_0(t) =\ \sqrt{\widetilde{H}(t) q_0(t)}$, i.e. the initial volume flow before employing the tidal turbines.

This yields the benchmarks $P_0 := \varrho g b\, \widetilde{H}(t) q_0(t)$ and $W_{avail} \coloneqq \varrho g b \int_0^T \widetilde{H}(t) q_0(t) \mathrm{d}t$, i.e. the initial power transport and available energy per period respectively.

Using the initial flow as benchmark, the optimal control strategy writes: The tidal turbines in a channel with neglectable hydraulic impedance and capacity are operated at best if the power extraction $P_T(t)$ results in a reduction of the channel flow $q(t)$ to $1/\sqrt{3}$ of the unaffected flow $q_0(t)$. The corresponding turbine power is $\eta\, 2\sqrt{3}/9$ of the initial power transport P_0. Thus, even without turbine and mixing losses ($\eta = 1$) only $2\sqrt{3}/9 \approx 38.5\ \%$ of the initially transported energy per period can be harvested.

The control strategy and upper limits remain valid even if additional losses have to be taken into consideration as long as they are proportional to q^2.

References

[1] P.F. Pelz, M. Metzler, *Free surface influence on the power extraction from tidal currents*, under consideration for publication in Journal of Fluid Mechanics (2016).

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