Designing a Feedback Control System via Mixed-Integer Programming

Lena C. Altherr, Thorsten Ederer, Ulf Lorenz, Peter Pelz, Philipp Pöttgen

Abstract Pure analytical or experimental methods can only find a control strategy for technical systems with a fixed setup. In former contributions we presented an approach that simultaneously finds the optimal topology and the optimal open-loop control of a system via Mixed Integer Linear Programming (MILP). In order to extend this approach by a closed-loop control we present a Mixed Integer Program for a time discretized tank level control. This model is the basis for an extension by combinatorial decisions and thus for the variation of the network topology. Furthermore, one is able to appraise feasible solutions using the global optimality gap.

Acknowledgements This work is partially supported by the German Federal Ministry for Economic Affairs and Energy funded project "Entwicklung hocheffizienter Pumpensysteme" and by the German Research Foundation (DFG) funded SFB 805.

1 Introduction

Conventional methods can only find an optimal control strategy for technical systems with a fixed setup. Another system topology may enable an even better control strategy. To find the global optimum, it is necessary to find the optimal control strategy for a wide variety of setups, not all of which might be obvious even to

Lena C. Altherr
Chair of Fluid Systems, TU Darmstadt e-mail: lena.altherr@fst.tu-darmstadt.de

Thorsten Ederer
Discrete Optimization, TU Darmstadt e-mail: ederer@mathematik.tu-darmstadt.de

Ulf Lorenz
Chair of Fluid Systems, TU Darmstadt e-mail: ulf.lorenz@fst.tu-darmstadt.de

Peter F. Pelz
Chair of Fluid Systems, TU Darmstadt e-mail: peter.pelz@fst.tu-darmstadt.de

Philipp Pöttgen
Chair of Fluid Systems, TU Darmstadt e-mail: philipp.poettgen@fst.tu-darmstadt.de

an experienced designer. We introduce a Mixed Integer Linear Program (MILP) for a feedback control that can be extended by combinatorial decisions. This approach is the basis for finding the optimal topology of a system and its optimal closed loop control simultaneously.

One example for a fluid system with different topological and control strategy options is a tank level control system. Pumps can either be used or not and they can be connected in series or in parallel. To control the water level, the rotational speed of the used pumps has to be adjusted. The goal may either be energy efficiency or a controller property like settling time or stability.

The tank level control system can be represented by a control circuit. To accurately optimize a feedback control system one needs to account for the time-dependent behavior of its components: P (proportional), I (integration) and D (derivation), PT1, PT2 (delay of first or second order) or PTt (dead-time). We discretized the time dependence and obtained a mixed-integer formulation based on a time-expanded flow network. This formulation allows us to appraise feasible solutions using the global optimality gap. Furthermore, we can combine this approach with a variation of the network topology [4].

2 State of the Art

A proportional-integral-derivative (PID) controller is one of the most commonly used closed-loop feedback methods. The control output $u(t)$ is computed from the present error P, the accumulated past errors I, and a prediction of future errors based on the current error derivative D [2]. The corresponding controller parameters k_p, k_i and k_d affect the variation of the error $e(t)$ in time. Adjusting these parameters is called tuning. For instance, with a bad parameter assignment the process variable may oscillate around a given set point, but given a good assignment the controller is fairly robust against small disturbances. However, tuning a controller for specific process requirements is highly nontrivial.

One approach to finding controller parameters is manual tuning. In this case, the proportional parameter k_p is raised until the control variable oscillates. This current value of k_p is then halved to obtain a good assignment. From this starting point, the other controller parameters can be derived [7]. Of course, this approach is rather error-prone and can be improved by computer-aided methods, e.g. by parameter optimization or evolutionary algorithms. However, each of these methods has at least one significant shortcoming: Discrete decisions like topology variations are not possible, or local optima cannot be distinguished from global optima.

MILP [5] is a modeling and solving technique for computer-aided optimization of real-world problems. Its main advantage compared to other optimization methods is that global optimality can be proven. During the last two decades, its application has become increasingly popular in the Operations Research community, e.g. to model problems in logistics, transportation and production planning. Commercial solvers are nowadays able to solve plenty of real-world problems with up to millions of variables and constraints. The field of applying mathematical optimization to physical-technical systems is called Technical Operations Research (TOR) [3].

In previous work we have shown how to optimize a hydrostatic power transmission system via mixed-integer programming [1]. The solution is a system topology with a corresponding open-loop control strategy. A weakness of this approach is that the actuating variable has to be known in advance for each time step. It is not possible to react to the current state of the fluid system. To overcome this, we include a mathematical description of closed-loop controllers into a mixed integer program. The aim of this paper is to illustrate the first step of our current work by presenting a small example.

3 Tank Level Control

We look at the example of a water level control. A tank is filled with water and has an outlet at the bottom out of which a certain amount of water drains, depending on the height of the stored water column. To fill the tank, a pump has to be activated. The filling speed depends on the pressure built up and the volume flow provided by the pump. Fig. 1 shows an exemplary setup.

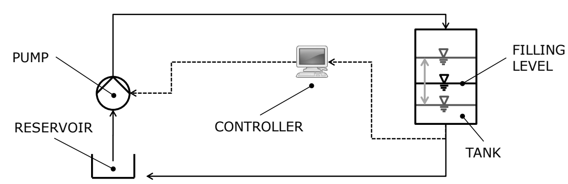

Fig. 1 A tank level control system. A pump conveys water from a reservoir into the tank while water is draining out of it. The rotary speed of the pump is set via controller. The task is to reach and maintain a given filling level in minimum time.

We model this setup by a mathematical graph $G = (V,E)$. The reservoir and the tank are represented by vertices V, pumps and pipes by edges E. The following physical equations and models describe the filling level system and are included into the MILP formulation. First, the continuity equation for incompressible fluids results in the conservation of volume flows Q.

$$\forall t \in T \; \forall v \in V : \quad \sum_{(i,v)\in E} Q_t^{i,v} = \sum_{(v,j)\in E} Q_t^{v,j} \tag{1}$$

The increase or decrease of the tank's filling volume V is approximated in our model by a sequence of static flows over short time intervals Δt.

$$\forall t \in T \setminus \{0\} : \quad V_t = V_{t-1} + \left(\sum_{(i,\text{tank})\in E} Q_t^{i,\text{tank}} - \sum_{(\text{tank},j)\in E} Q_t^{\text{tank},j} \right) \cdot \Delta t \tag{2}$$

If a pump is activated (indicated by the binary variable a_t^{ij}), the pressure difference between both connectors i and j is fixed to Δp_t^{ij}, which is given by the pump characteristic, cf. equation (10). Otherwise, the pressure levels of the two connectors are decoupled by means of a big-M formulation.

$$\forall t \in T \; \forall (i,j) \in \text{Pumps}: \quad |p_t^j - p_t^i| \leq \Delta p_t^{ij} + M \cdot (1 - a_t^{ij}) \tag{3}$$

In an ideal pipe and in case of turbulent flow, the pressure loss is described by an origin-rooted parabola. In real systems, an interference due to dynamic effects can be observed. Still, the pressure loss can be well fitted by a general quadratic form.

$$\forall t \in T \; \forall (i,j) \in \text{Pipes}: \quad p_t^j - p_t^i = c_2 \cdot Q_t^{ij,\text{sqr}} + c_1 \cdot Q_t^{ij} + c_0 \tag{4}$$

We have to introduce an auxiliary variable for the squared flowrate and a piecewise linearization of the square function. For univariate functions, the incremental method was shown to be very efficient [6]. With progress variables $\delta \in [0,1]$ on linearization intervals D and passing indicators $z \in \{0,1\}$, this results in:

$$\forall t \in T \; \forall (i,j) \in E \; \forall (g,h) \in D: \quad z_{t,g}^{ij} \geq \delta_{t,gh}^{ij} \geq z_{t,h}^{ij} \tag{5}$$

$$\forall t \in T \; \forall (i,j) \in E: \quad Q_t^{ij} = \sum_{(g,h) \in D} (\tilde{Q}_{t,h}^{ij} - \tilde{Q}_{t,g}^{ij}) \cdot \delta_{t,gh}^{ij} \tag{6}$$

$$\forall t \in T \; \forall (i,j) \in E: \quad Q_t^{ij,\text{sqr}} = \sum_{(g,h) \in D} (\tilde{Q}_{t,h}^{ij,\text{sqr}} - \tilde{Q}_{t,g}^{ij,\text{sqr}}) \cdot \delta_{t,gh}^{ij} \tag{7}$$

The pump characteristic is a nonlinear relation – caused by the pump geometry – between its rotational speed n, flowrate Q, pressure boost Δp and power input P. This dependence is MILP-representable by a convex combination formulation [6] with weights $\lambda \in [0,1]$ on nodes K and a selection $\sigma \in \{0,1\}$ of simplices X.

$$\forall t \in T: \quad \sum_{x \in X} \sigma_{t,x}^{\text{pump}} = \sum_{k \in K} \lambda_{t,k}^{\text{pump}} = a_t^{\text{pump}} \tag{8}$$

$$\forall t \in T \; \forall k \in K: \quad \lambda_{t,k}^{\text{pump}} \leq \sum_{x \in X(k)} \sigma_{t,x}^{\text{pump}} \tag{9}$$

$$\forall t \in T: \quad n_t^{\text{pump}} = \sum_{k \in K} \lambda_{t,k}^{\text{pump}} \cdot \tilde{n}_k^{\text{pump}}, \quad \Delta p_t^{\text{pump}} = \sum_{k \in K} \lambda_{t,k}^{\text{pump}} \cdot \Delta \tilde{p}_k^{\text{pump}} \tag{10}$$

$$\forall t \in T: \quad Q_t^{\text{pump}} = \sum_{k \in K} \lambda_{t,k}^{\text{pump}} \cdot \tilde{Q}_k^{\text{pump}}, \quad P_t^{\text{pump}} = \sum_{k \in K} \lambda_{t,k}^{\text{pump}} \cdot \tilde{P}_k^{\text{pump}} \tag{11}$$

The pressure at the bottom of the tank is proportional to the water column height, with the tank's area A, the density of water ρ and the gravitational acceleration g.

$$\forall t \in T: \quad p_t^{\text{tank}} = \rho \cdot g \cdot \frac{V_{t-1}^{\text{tank}}}{A} \tag{12}$$

Additional Variables and Constraints for the Control Circuit

We model a PI controller with the equation

$$n^{\mathrm{ref}}(t) = k_p \cdot \Delta V(t) + k_i \cdot \int_{\tau \le t} \Delta V(\tau)\, \mathrm{d}\tau \tag{13}$$

where k_p and k_i are control parameters that determine if and how fast the control deviation converges to zero. Discretizing this relation yields input variables ΔV_t for the proportional and $S_t^V = \sum_{\tau \le t} V_\tau$ for the integral part.

$$\forall t \in T: \quad n_t^{\mathrm{ref}} = n_t^p + n_t^i, \quad n_t^p = k_p \cdot \Delta V_t, \quad n_t^i = k_i \cdot S_t^V \tag{14}$$

$$\forall t \in T: \quad \Delta V_t = V^{\mathrm{ref}} - V_t^{\mathrm{tank}}, \quad S_t^V = S_{t-1}^V + \Delta V_t \tag{15}$$

The products of continuous variables $k_p \cdot \Delta V_t$ and $k_i \cdot S_t^V$ are linearized by the convex combination method, cf. equations (8) and (10).

If the reference value is larger than the pump's maximum rotational speed or smaller than the pump's minimum rotational speed, it cannot be reached by the actual value. We introduce two auxiliary variables Ω_t and ω_t for each time interval that peak if and only if the reference value is out of bounds.

$$\Omega_t \cdot (n_{\max}^p + n_{\max}^i - n_{\max}) \ge n_t^{\mathrm{ref}} - n_{\max} \ge (1 - \Omega_t) \cdot (n_{\min}^p + n_{\min}^i - n_{\max}) \tag{16}$$

$$\omega_t \cdot (n_{\min}^p + n_{\min}^i - n_{\min}) \le n_t^{\mathrm{ref}} - n_{\min} \le (1 - \omega_t) \cdot (n_{\max}^p + n_{\max}^i - n_{\min}) \tag{17}$$

If the out-of-bounds indicators are deactivated, the rotational speed of the pump has to equal the output $n_p + n_i$ of the PI controller. If an out-of-bound indicator peaks, the rotational speed has to reach its corresponding bound.

$$\omega_t \cdot (n_{\min}^p + n_{\min}^i - n_{\max}) \le n_t^{\mathrm{ref}} - n_t \le \Omega_t \cdot (n_{\max}^p + n_{\max}^i - n_{\min}) \tag{18}$$

$$n_{\max} - n_t \le (n_{\max} - n_{\min}) \cdot (1 - \Omega_t) \tag{19}$$

$$n_{\min} - n_t \ge (n_{\min} - n_{\max}) \cdot (1 - \omega_t) \tag{20}$$

We want to minimize the time m until the tank's volume stays within a given error bound ε_V. We model this using a big-M formulation and a monotonically increasing binary function $s_t \in \{0,1\}$ with peak indicator $j_t^s \in \{0,1\}$.

$$\forall t \in T: \quad |\Delta V_t| \le \varepsilon_V + M \cdot (1 - s_t), \quad s_t - s_{t-1} = j_t^s \tag{21}$$

$$\sum_{t \in T} j_t^s = 1, \qquad \text{minimize} \quad \sum_{t \in T} t \cdot j_t^s \tag{22}$$

Fig. 2 shows the result of a test instance. The MILP model needs 71 s to reach the target volume. In a detailed simulation with the controller parameters obtained from the optimization, the desired set point is reached within approximately 75 s. In contrast to the optimization model, no oscillation of the control variable is observed.

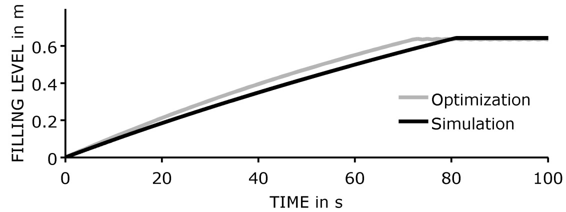

Fig. 2 Comparison of optimization result and simulation. The step size is set to 1 s in our model. The simulation result is computed with the DASSL solver with variable order and a time step of 0.02 s.

The approximation used in the MILP is comparable to the Euler method, a first-order method with an error proportional to the step size. The computation times are still rather long, ranging from minutes with MILP start to days without. For ongoing research it might be interesting to investigate problem-specific primal heuristics.

4 Conclusion

We motivated a MILP formulation for the design process of a closed-loop control. We started from a model for finding an optimal open-loop control for a filling level control system and extended it to find an optimal closed-loop control. The advantage of this formulation is that the load does not need to be known in advance to adjust the rotational speed of the pump. Instead, the implemented controller is able to compute the actuating variable in realtime based on the current pressure head of the tank. Owing to the mixed-integer formulation one can easily extend our model by integer or binary variables. We aim to include discrete purchase decisions or network topology variations in future projects.

References

[1] B. Dörig, T. Ederer, P. Hedrich, U. Lorenz, P. F. Pelz, and P. Pöttgen. Technical operations research (TOR) exemplified by a hydrostatic power transmission system. 2014.
[2] C. Kilian. Modern control technology. thompson delmar learning. *Electronic Engineering*, 2005.
[3] P. Pelz and U. Lorenz. Besser geht's nicht! TOR plant das energetisch optimale Fluidsystem. *delta p, Ausgabe 3, ErP Spezial*, 2013.
[4] P. F. Pelz, U. Lorenz, T. Ederer, S. Lang, and G. Ludwig. Designing pump systems by discrete mathematical topology optimization: The artificial fluid systems designer (AFSD). 2012.
[5] A. Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, 1998.
[6] J. P. Vielma, S. Ahmed, and G. Nemhauser. Mixed-integer models for non-separable piecewise-linear optimization: Unifying framework and extensions. *Operations research*, 58(2):303–315, 2010.
[7] J. Ziegler and N. Nichols. Optimum settings for automatic controllers. *trans. ASME*, 64(11), 1942.