Journal of Biomechanics 47 (2014) 3415–3422

Contents lists available at ScienceDirect

Journal of Biomechanics

journal homepage: www.elsevier.com/locate/jbiomech
www.JBiomech.com

Validated biomechanical model for efficiency and speed of rowing ☆

Peter F. Pelz *, Angela Vergé

Technische Universität Darmstadt, Germany, Chair of Fluid Systems, Magdalenenstr. 4, D-64289 Darmstadt, Germany

A R T I C L E I N F O

Article history:
Accepted 27 June 2014

Keywords:
Speed of rowing
Kleiber's law
Allometric scaling
Geometric scaling
Similarity
Froude propulsion efficiency
Theoretical upper limit for repetitive motion

A B S T R A C T

The speed of a competitive rowing crew depends on the number of crew members, their body mass, sex and the type of rowing—sweep rowing or sculling. The time-averaged speed is proportional to the rower's body mass to the $1/36^{\text{th}}$ power, to the number of crew members to the$1/9^{\text{th}}$ power and to the physiological efficiency (accounted for by the rower's sex) to the $1/3^{\text{rd}}$ power. The quality of the rowing shell and propulsion system is captured by one dimensionless parameter that takes the mechanical efficiency, the shape and drag coefficient of the shell and the Froude propulsion efficiency into account. We derive the biomechanical equation for the speed of rowing by two independent methods and further validate it by successfully predicting race times. We derive the theoretical upper limit of the Froude propulsion efficiency for low viscous flows. This upper limit is shown to be a function solely of the velocity ratio of blade to boat speed (i.e., it is completely independent of the blade shape), a result that may also be of interest for other repetitive propulsion systems.

1. Introduction

The speed of a competitive rowing crew depends on the number of crew members, their body mass, sex and the type of rowing—sweep rowing or sculling. Since there are boats with 1, 2, 4, or 8 crew members there are in theory 32 rowing classes as shown schematically in Fig. 1.

Our objective is to derive and validate a general biomechanical equation for the time-averaged speed of rowing by taking into account the metabolic rate of the rower and all relevant loss mechanisms. The equation accounts for the number of crew members n (i.e., the number of prime movers), the body mass of the competitive rower m, the propulsion type (either sweep rowing or sculling), and the gender of the rowing crew. The literature survey is presented according to the four possible approaches that are found there: (1) empirical research; (2) detailed physical modeling; (3) time-averaged energy balance in integral form; and (4) dimensional analysis.

We use the independent approaches (3) and (4) to derive the same equation for the speed of rowing, whereas most research on the biomechanics of rowing has followed the first two approaches (Affeld, et. al., 1993; Kleshnev, 1999; Cabrera, et. al., 2006). There is no research on the influence of body mass on the boat speed. A dimensional analysis (McMahon, 1971) that takes the geometric similarity of the shells for different rowing classes into account yields the $1/9^{\text{th}}$ power law for the relation between speed and number of rowers. For a dimensional analysis, it is typical only to consider continuous scales and magnitudes. In contrast, we apply the methods of dimensions, as Rayleigh described it (Rayleigh, 1877), to discrete variables, such as the number n of crew members. Here, we define a discrete measure N and require scale invariance, i.e. Bridgman's postulate (Bridgman, 1922), also for the number of crew members (Barenblatt, 2003). Although rowing is a biomechanical system, the prime mover (i.e. the oarsmen and, in particular, his or her weight), has been thus far left out of the equation. It is one of our main objectives to determine the dependency of the body mass of the competitive rowers on the average boat speed. The influence of the mass is twofold. On the one hand, due to allometric scaling, the input power increases with body mass to the power of $3/4^{\text{th}}$ according to Kleiber's law (Kleiber, 1932 and 1975) and, in fact, we validate Kleiber's law for heavyweight and lightweight, male and female crews. On the other hand, as body mass increases, the shell surface increases due to Archimedes' law. This results in an increase in frictional drag. As will be seen, for a competitive rower, boat speed increases with body mass only to the power of $1/36^{\text{th}}$.

The outline of the paper is as follows. We first derive the equation for the speed of rowing and show the scale invariance of rowing. The identical result is then achieved independently by means of an energy balance for rowing that is analogous to Lighthill's analysis of fish movement (Lighthill, 1960). In the Section 3 "Validation of Allometric Scaling of Rowing", we confirm Kleiber's law by analyzing race times of winning crews at world

☆ *Dedicated to Professor Dieter Hellmann – a former oarsman – on the occasion of his 70th birthday.*

* Corresponding author. Tel.: +49 6151 162153; fax: +49 6151 162453
E-mail address: peter.pelz@fst.tu-darmstadt.de (P.F. Pelz).

http://dx.doi.org/10.1016/j.jbiomech.2014.06.037
0021-9290/

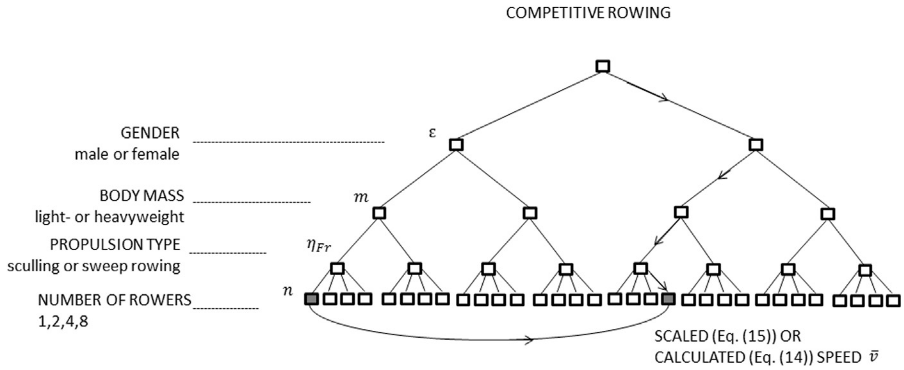

Fig. 1. There are in theory 32 possible rowing classes (neglecting cox) which differ in gender, body mass, propulsion type, number of crew members resulting in different average boat speed.

cup and Olympic regattas. In the final section 4 we use our results to predict the boat speed of various rowing classes at major regattas and validate the predictions.

2. Scale invariance and speed of rowing

There are three types of similarity and scaling: (1) geometric similarity/scaling; (2) physical similarity/scaling; and (3) allometric similarity/scaling. Geometric similarity is a special form of physical similarity based on Bridgman's postulate (Bridgman, 1922). Kleiber's law (Kleiber, 1932 and 1975), i.e. the metabolic rate and hence the mechanical power of an organism is proportional to its body mass raised to the power of 3/4$^{\text{th}}$ is an allometric scaling. This empirical relationship has been found to hold across the living world from bacteria to blue whales. Today, it is agreed that the 3/4$^{\text{th}}$ power can be explained by the specific geometry (West, et. al., 1999) or type (Banavar, et. al., 1999) of the metabolic networks, such as the geometry and topology of blood vessels. As far as we know there is no research on allometric scaling in rowing. As will be seen, combining all three scaling and similarity methods will lead us to the desired speed of rowing.

McMahon, (1971) was the first to notice the geometric similarity in rowing. For a shell, the most relevant geometric data are boat length L, surface area $A := \kappa_A^2 L^2$, shell width $B := \kappa_B L$, and displaced water volume $V := \kappa_V^3 L^3$. The shape factors $\kappa_A, \kappa_B, \kappa_V$ are given in Table 1 for 23 different rowing classes. The shape factors show a maximum relative standard deviation of 3.7%. The first requirement of any shell is to support the weight of the crew. This is described by a force balance of the buoyancy (proportional to the water density ϱ) and the total weight of crew, cox, and shell according to Archimedes' principle. The ratio of the crew's total weight without cox to the overall weight including cox and boat, defined by $\kappa_M := nm/\varrho V$ is 19%, with a relative standard deviation of 20%. This large variance is of only minor importance, since the dimensionless, volume-specific surface of the shell, defined by

$$\kappa := \frac{A}{(V\kappa_M)^{2/3}} = \frac{\kappa_A^2}{\kappa_V^2 \kappa_M^{2/3}}, \tag{1}$$

fully accounts for the shell's drag. It is equal to 12.09, with a relative standard deviation of only 4.9%. Since κ is nearly constant, the surface area scales to the power 2/3 with the number of oarsmen and body mass:

$$A = \kappa (V\kappa_M)^{2/3} = \kappa (nm/\varrho)^{2/3}. \tag{2}$$

Along with the *geometric similarity*, there is a *physical similarity* for the shell's drag for all rowing classes: The time-averaged drag force of the shell $\overline{D}$ (time average is indicated by an over bar) depends on the speed of rowing $\bar{v}$ relative to calm water, the density ϱ and kinematic viscosity ν of water, the specific gravity constant g, the stroke rate $1/\tau$, and the shape of the shell, which is the same for all rowing classes due to the discussed geometric similarity: $\overline{D} = \overline{D}(\varrho, \bar{v}, A, \nu, g, \tau, shape)$. This equation remains the same, regardless of which fundamental units (Rayleigh, 1877; Bridgman, 1922) are used to express the 7 quantities. Since this is a dynamic problem, the fundamental dimensions of length, mass, and time $[LMT]$, or their equivalents length, force, and time $[LFT]$, are used preferentially. Due to the required scale invariance, the relation can be expressed equivalently using only 4 dimensionless parameters:

$$\bar{c}_D = \bar{c}_D(\overline{Re},\ \overline{Fr},\ \lambda, shape), \tag{3}$$

where $\bar{c}_D := 2\overline{D}/(\varrho \bar{v}^2 A)$ is the drag coefficient, $\overline{Re} := \bar{v}L/\nu$ is the Reynolds number, $\overline{Fr} := \bar{v}/\sqrt{gL}$ is the Froude number, and $\lambda := \bar{v}/u = \bar{v}/\Omega l$ ($\Omega = 2\pi/\tau$, outboard length l) is the dimensionless boat speed known as advance ratio (Newmann, 1977). The drag coefficient is nearly constant for all rowing classes $(2.65 \pm .15) \times 10^{-3}$ (Mang, 2008). Thus, competitive rowing hulls exhibit not only *geometrical similarity*, but also an approximate *physical similarity*.

For the moment, we assume (to be validated in Section 3) that the mechanical power $\overline{P}_0$ of competitive rowers—whether lightweight or heavyweight, male or female—scales with their body masses m according to Kleiber's allometric scaling law (Kleiber, 1932 and 1975)

$$\overline{P}_0 = \varepsilon\, m^{3/4}\,. \tag{4}$$

Since ε is a measure of the physiological quality of the rower, it is justified to name it *physiological efficiency*. Since $\overline{P}_0$ and m are rower-specific physical quantities, we introduce a *dimension N* to account for the number of oarsmen n. The *scale* can either be one rower or two rowers, counting in *units* of one, two, or more. Hence, together with the dimensions for dynamic systems length L, mass M, and time T, the suitable fundamental system of dimensions is $[LMTN]$. With the above-discussed physical similarity accounting for the shell's drag, the speed of rowing can be reduced to a function of the following dimensional quantities: The physiological efficiency ε and body mass m of the rower, the water density ϱ, and the number of crew members n:

$$\bar{v} = \bar{v}(\varepsilon,\ m,\ \varrho,\ n). \tag{5}$$

This equation must remain unchanged, regardless of the fundamental units used to express the five quantities. Bridgman's postulate, i.e. “the absolute meaning of relative quantities” (Bridgman, 1922), results in the scale invariance of Eq. (5). Hence, $\bar{v} = \bar{v}(\varepsilon,\ m,\ \varrho,\ n)$ is equivalent to a single dimensionless product $\Pi = const$. The dimension of speed is given by $[\bar{v}] = L^1 T^{-1}$, that of density, by $[\varrho] = M^1 L^{-3}$. Since mass and physiological efficiency

Table 1
Shell data for different rowing classes and the associated dimensionless products (Mang, 2008).

	Number of			Weight per oarsman	Shell data					Dimensionless products				
											Shape			
	Oarsman	Blades			Lenght	Width	Surface	Volume	Mass	Boat mass/Oarsman mass				
	n	i		m in kg	L in m	B in m	A in m²	V in m³	m_B in kg	$1\text{-}\kappa_M$	κ_A $A^{1/2}/L$	κ_B B/L	κ_V $V^{1/3}/L$	κ
Men	1	2	ca.	95	8.33	0.29	2.39	0.109	14	15%	19%	3.5%	5.7%	11.64
	2	2	ca.	95	9.98	0.35	3.69	0.217	27	14%	19%	3.5%	6.0%	11.31
	2	1	ca.	95	9.95	0.36	3.68	0.217	27	14%	19%	3.6%	6.0%	11.30
	4	2	ca.	95	12.89	0.46	5.92	0.432	52	14%	19%	3.6%	5.9%	11.42
	4	1	ca.	95	12.89	0.46	5.90	0.430	50	13%	19%	3.6%	5.9%	11.38
With cox (55 kg)	8	1	ca.	95	17.63	0.56	10.05	0.911	96	20%	18%	3.2%	5.5%	12.39
Men light-weight	1	2		72.5	7.78	0.27	2.03	0.084	14	19%	18%	3.5%	5.6%	12.19
	2	2		70	9.40	0.33	3.14	0.167	27	19%	19%	3.5%	5.9%	11.95
	2	1		70	9.40	0.33	3.14	0.167	27	19%	19%	3.5%	5.9%	11.95
	4	2		70	11.78	0.43	4.96	0.332	52	19%	19%	3.7%	5.9%	11.86
	4	1		70	11.78	0.43	4.94	0.330	50	18%	19%	3.7%	5.9%	11.80
With cox (55 kg)	8	1		70	16.85	0.56	8.33	0.656	96	27%	17%	3.3%	5.2%	13.61
Women	1	2	ca.	73	7.78	0.27	2.06	0.087	14	19%	18%	3.5%	5.7%	12.11
	2	2	ca.	73	9.40	0.33	3.20	0.173	27	18%	19%	3.5%	5.9%	11.80
	2	1	ca.	73	9.60	0.34	3.23	0.173	27	18%	19%	3.5%	5.8%	11.92
	4	2	ca.	73	11.78	0.43	5.05	0.344	52	18%	19%	3.7%	5.9%	11.71
	4	1	ca.	73	11.78	0.43	5.03	0.342	50	17%	19%	3.7%	5.9%	11.66
With cox (55 kg)	8	1	ca.	73	16.85	0.56	8.79	0.730	96	26%	18%	3.3%	5.3%	13.24
Women light-weight	1	2		59	7.40	0.27	1.82	0.071	14	24%	18%	3.6%	5.6%	12.69
	2	2		57	9.00	0.33	2.82	0.141	27	24%	19%	3.7%	5.8%	12.48
	2	1		57	9.00	0.33	2.82	0.141	27	24%	19%	3.7%	5.8%	12.48
	4	2		57	11.78	0.43	4.55	0.280	52	23%	18%	3.7%	5.6%	12.64
	4	1		57	11.78	0.43	4.54	0.278	50	22%	18%	3.7%	5.5%	12.56
Mean										19%	19%	3.5%	5.7%	12.09
Standard deviation										3.8%	0.5%	0.1%	0.2%	0.59
Rel. standard deviation										20%	2.8%	3.6%	3.7%	4.9%

are specific data for one rower, we obtain the dimensions $[m]=M^1N^{-1}$ and from (4)

$$[\varepsilon]=\frac{[\overline{P}_0]}{[m^{3/4}]}=\frac{ML^2/(T^3N)}{(M/N)^{3/4}}=M^{1/4}\,L^2\,T^{-3}N^{-1/4}\,.$$

Thus the dimension matrix reads

	$\overline{v}$	ε	m	ϱ	n
L	1	2	0	−3	0
M	0	1/4	1	1	0
T	−1	−3	0	0	0
N	0	−1/4	−1	0	1

Because it has a rank of 4, the single dimensionless product is derived

$$\Pi=\frac{\overline{v}\varrho^{1/9}}{\varepsilon^{1/3}\,m^{1/36}n^{1/9}}=const. \tag{6}$$

This result already tell us the required dependence on the rower's physiological quality, which is given by the body mass and physiological efficiency. There is an increase in boat speed with body mass to the power of 1/36 only. That is, a crew with an average body mass of 95 kg is only 0.85% faster than a crew with an average body mass of 70 kg when rowing in the same class with the same physiological efficiency.

We shall now derive Eq. (6) independently, using a time-averaged energy balance (cf. Appendix A). This will allow us to corroborate the validity of Eq. (6) and obtain the so far unknown constant in the equation. As will be seen, it is determined solely by the quality of the hull and the propulsion system (oars and blades). Hence there are three types of quality to be considered, as shown in Fig. 2: (1) the physiological quality of the rower, as given by her/his physiological efficiency and body mass; (2) the quality of the hull; and (3, 4) the quality of the propulsion system. The quality of the hull is determined by κ, $\overline{c}_D$, and η_M–the mechanical efficiency of the shell. The quality of the propulsion system is determined by Froude's propulsion efficiency η_{Fr} (Newmann, 1977). We will now discuss these two efficiencies η_M and η_{Fr}.

First: Although they are important for any repetitive propulsion system, we have thus far ignored mechanical losses within the structure of the boat itself. The power $\overline{P}_S$ applied to the water is less than the power $n\overline{P}_O$ applied by the n oarsmen by an amount equal to the mechanical loss E per stroke. As soon as a periodic force time history $F(t)=F(t+\tau)$ with its maximum $\hat{F}$ is applied to the blades and an equal reaction force to the structure, elastic energy E is stored within the structure. Here, $E\approx\hat{F}^2/2k\approx2\overline{D}^2/k$ for $\overline{D}=\int_0^1F(t)\,dt/\tau\approx\beta\hat{F}/2$, where k denotes the structural stiffness. This energy is completely dissipated by a damped, free oscillation of the system (shell, rigger and oar) when the rower releases the force on the blade at the end of each stroke. Hence, the mechanical efficiency, a dimensionless measure of dissipation, is given by

$$\eta_M:=\frac{\overline{P}_S}{n\overline{P}_O}=1-\frac{E}{\tau\overline{P}_O}\approx1-\eta_M\frac{2}{\beta^2}\frac{\overline{D}^2}{\tau\overline{P}_Sk}\approx1-\eta_M\eta_{Fr}\frac{2}{\beta^2}\frac{\overline{D}}{\tau\overline{v}k},$$

for $\overline{P}_S\eta_{Fr}\approx\overline{v}\overline{D}$ (see Fig. 2 and Eq. (8)). With the drag coefficient (Eq. (3)) this leads to:

$$\eta_M=\left(1+\frac{\eta_{Fr}}{\beta^2}\frac{\overline{c}_D}{k_+}\right)^{-1}\approx\left(1+\frac{\overline{c}_D}{k_+}\right)^{-1},\;k_+:=\frac{k}{\varrho A\overline{v}/\tau}. \tag{7}$$

Hence, most important for the mechanical losses due to the repetitive motion is the dimensionless stiffness given by $k_+:=k/(\varrho A\,\overline{v}/\tau)$. For lightweight structures, the product of weight and deflection due to an applied force is most important. Deflection and weight scale with shell thickness to the power of −3 (Kirchhoff plate) and 1, respectively. Hence, heir products scale to the power of −2. Increasing the shell thickness with honeycomb material, for example, is the most effective way to increase hull stiffness and, hence, the mechanical efficiency of rowing. In fact replacing wood with carbon reinforced composite and honeycomb material for oars and shell significantly increased the mechanical efficiency and hence the boat speed at major regattas.

Second: The objective of any propulsion system is to develop a positive thrust force T to overcome drag. If the system moves at a constant speed v, the thrust force is in balance with the drag and $T=-D$. For a repetitive motion with acceleration $a(t)=a(t+\tau)$ and cycle time $\tau=1/$stroke rate, this is also true in the time average (i.e., $\overline{T}=-\overline{D}$), since the time average of the acceleration vanishes (i.e., $\overline{a(t)}\equiv0$) due to periodicity. Hence, the beneficial power for rowing, as for any propulsion system, is $\overline{P}_{use}=\overline{vT}$. The boat is propelled by n oarsmen. All oarsmen together apply the work $\overline{P}_S$ per unit of time to the water by means of one or two blades ($i=1$ one for sweep rowing, $i=2$ for sculling). To produce the vortex wake of the blades and the additional dissipation, the input power $\overline{P}_S$ is greater than $\overline{P}_{use}=\overline{vT}$ by the amount $\overline{P}_{loss}$ (Lighthill, 1960). Hence, the Froude propulsive efficiency is defined, as is usual in marine hydrodynamics (Newmann, 1977) and biomechanics (Lighthill, 1960; Purcell, 1997), by

$$\eta_{Fr}:=\frac{\overline{vT}}{\overline{P}_S}=\frac{1}{1+(\overline{P}_{loss}/\overline{vT})}. \tag{8}$$

There are two hydrodynamic losses associated with the propulsion. One is due to the vortex wake of the blade, which is

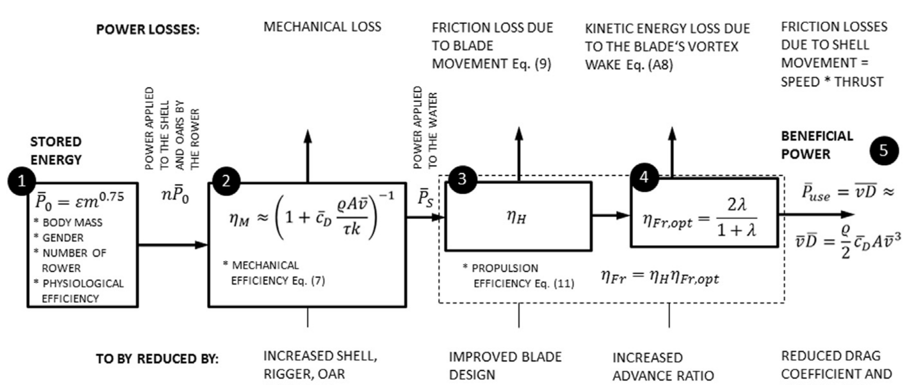

Fig. 2. Energy conversion from the stored energy to the usable power with all power losses in rowing. The friction losses due to blade movement, i.e. η_H, and the drag coefficient $\overline{c}_D$ may be determined by CFD analysis, which is not necessary in the context of this work, since η_H is "measured" applying Eqs. (14) and (15) to race times (see below). There are 5 main possibilities to increase boat speed. First: By increasing the rowers physiological efficiency ε. Second: By increasing the stiffness, rigger, oar stiffness k as it was done in the past by replacing wood with carbon reinforced composite and honeycomb structures. Third: By increasing the hydraulic efficiency of the blade as it was done by inventing an asymmetric blade shape. Fourth: By increasing the advance ratio. This is the main difference between sculling and sweep rowing. The advance ratio in sculling is higher compared to sweep rowing. Fifth: By reducing the drag coefficient and surface area.

independent of the Reynolds number, and one is due to viscous friction. The former dominates for high Reynolds numbers (Lighthill, 1960) (a typical Reynolds number for swimming or rowing is $Re_{BLADE} \sim 10^4$ (Purcell, 1997)), whereas the latter dominates for low Reynolds numbers (a typical Reynolds number for a flagellate single cell is about 10^{-3} (Purcell, 1997)). Since flow in the high Reynolds number regime is also called "ideal fluid motion" (Newmann, 1977), we write $\overline{P}_{loss} = \overline{P}_{ideal} + \overline{P}_{viscous}$. As shown in the Appendix for non-rotational (i.e. potential) flow, the first loss equals the rate of change of the kinetic energy of the fluid $\overline{P}_{ideal} = \overline{DK}/Dt$ (i.e. the power of the pressure forces). The dissipation rate, or viscous loss, equals the rate of change of the internal energy $\overline{P}_{viscous} = \overline{DE}/Dt$ (Spurk, et. al., 2008). With this split, we can write the propulsion efficiency as a product of hydraulic efficiency and optimal Froude propulsion efficiency: $\eta_{Fr} = \eta_H\, \eta_{Fr,opt}$. The hydraulic efficiency η_H is defined as usual, that is, as a dimensionless measure of dissipation rate:

$$\eta_H(Re_{BLADE}, Fr_{BLADE}, blade\ shape) := 1 - \frac{\overline{P}_{viscous}}{\overline{P}_S}. \tag{9}$$

$\eta_{Fr,opt}$ is the upper limit of the Froude propulsion efficiency due to the vortex wake:

$$\eta_{Fr,opt}(\lambda) := \frac{1}{1 + (\overline{P}_{ideal}/\overline{vT})}. \tag{10}$$

As will be seen, $\eta_{Fr,opt}$ depends solely on the dimensionless boat speed $\lambda = \overline{v}/\Omega l$ and not at all on the blade shape.

Of course, it is possible to determine η_{Fr} by measuring force histories (Affeld, et. al., 1993; Kleshnev, 1999) (cf. Appendix A) or doing CFD simulations. However, to analyze the vortex wake of the blade sketched in Fig. 3 we proceed in a manner similar to Lighthill, (1960) in his study of slender fish swimming (regarded today as a milestone of biomechanics) and generate Eq. (10) in this way.

Typically, flow along a rowing blade results in flow separation (stall) and the influence of the free surface cannot be regarded as negligible. Often, there is no water at all on the lee side of the blade, an effect known as ventilation in marine hydrodynamics. In addition and most important, the flow is transient, which causes non-dissipative forces proportional to the added mass (Newmann, 1977). Both arguments call into question the application of quasi-static air-foil theory, as has been done in the past (Cabrera, et. al., 2006). Due to these difficulties, there is a clear advantage by analyzing the wake: its temporal and spatial structure is much simpler than the flow along the blade and these structures are interconnected by the momentum and energy balance (Taylor, 1953). Suppose $W_\infty \le u - \overline{v} - C_\infty$, with $u = \Omega l$, is the typical undisturbed approach velocity relative to the blade during one stroke. This typical velocity may be set equal to the maximal blade velocity relative to water ($C_\infty = 0$ for calm water; cf. Fig. 3d). The kinetic energy and momentum of the flow around the blade is hence $K' \sim W_\infty^2$ and $I' = dK'/dW_\infty$, respectively for an ideal fluid. Following Taylor (1953) "It seems that the redistribution of velocities involved in diffusing the vorticity within the core of the vortex ring is not likely to change the energy much or impulse at all, so that" $K = K'$, $I = I'$. Hence, $K = m' W_\infty^2/2$ is the kinetic energy of the blade's vortex wake (Taylor, 1953), with m' being the mass of the accelerated fluid, i.e. the so-called *added mass*. Due to the repetitive motion, the time-averaged energy loss caused by the vortex wake is given by K/τ. For the considered potential flow, the momentum of the blade's wake is $I = dK/dW_\infty = m' W_\infty$. Hence, the time-averaged thrust produced by one blade is I/τ. The required power ratio $\overline{P}_{ideal}/\overline{vT}$ in Eq. (10) is hence equal to the velocity ratio $W_\infty/(2\overline{v})$. The Froude propulsion efficiency (8) is thus derived for any repetitive motion (rowing, paddling, swimming, …) to be

$$\eta_{Fr} = \eta_H \eta_{Fr,opt} = \eta_H \frac{1}{1 + (\overline{K}/(\overline{v dK/dW_\infty}))} = \eta_H \frac{1}{1 + (1/2)(W_\infty/\overline{v})}. \tag{11}$$

Thus the Froude propulsion efficiency for the repetitive motion is similar to that of a screw propeller (cf. (Betz, 1959), p 194 and (Prandtl, 1944), p 199). With $W_\infty/\overline{v} \le (1-\lambda)/\lambda$ for calm water (cf. Fig. 3d), the ideal Froude propulsion efficiency is limited by

$$\text{max. propulsion efficiency of rowing}: \quad \eta_{Fr,opt} = \frac{2\overline{v}/u}{1 + \overline{v}/u} = \frac{2\lambda}{1+\lambda}, \text{ with } u = \Omega l. \tag{12}$$

Lighthill's result for the motion of eel-like fishes with a permanent, horizontal trailing edge (Lighthill, 1960) reads

$$\text{Anguilliform propulsion}: \quad \eta_{Fr,opt} = \frac{1 + \overline{v}/u}{2}, \tag{13}$$

with u being the speed of a traveling wave moving down the fish's body, as sketched in Fig. 4.

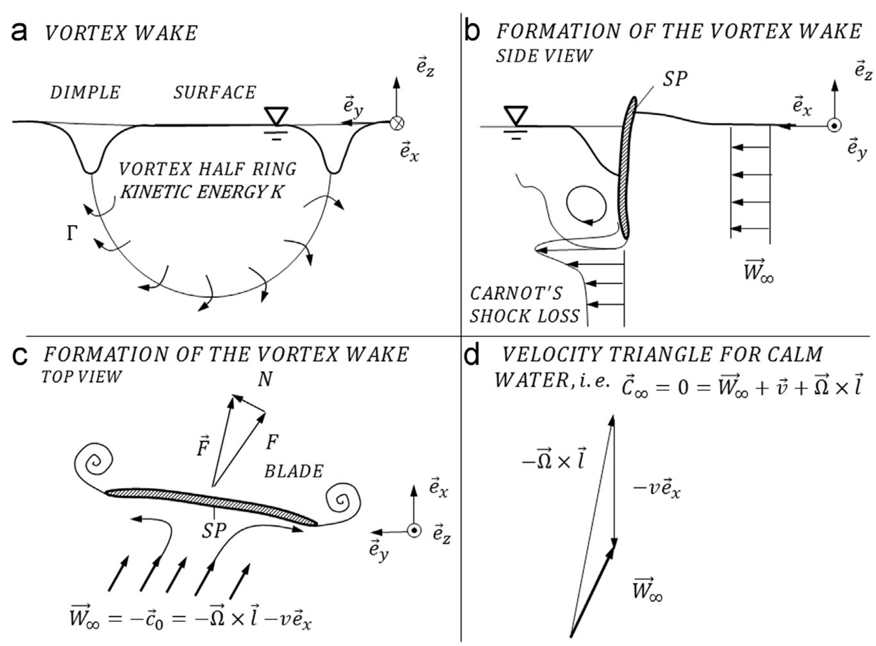

Fig. 3. (a) Vortex wake of the blade; (b) and (c) Formation of the vortex wake; and (d) Velocity triangle (outboard length l).

For rowing applies $u > \bar{v}$. Because of this the water is approaching the blade for a blade fixed observer as sketched in Fig. 3a. For competitive canoeing the situation is the opposite: $u < \bar{v}$. Hence the flow is approaching the blade from the opposite side as sketched in Fig. 3a. This demonstrates the importance of the above mentioned transient, i.e. accelerated motion of the blade: a canoe paddle can only produce a positive thrust through the blade's acceleration $\dot{u}$ together with the blades added mass, i.e. the accelerated fluid mass. The added mass is proportional to the fluid density ϱ and a typical blade length size b to the power of 3. The acceleration is proportional to the outboard length l times a typical angle $\hat{\varphi}$ divided by the square of the cycle time $\hat{\varphi}$: $m'\dot{u} \approx \varrho b^3 l\hat{\varphi}/\tau^2$. Hence the thrust is composed of three additive forces. First and most important: An acceleration term. The basilisk or Jesus Christ lizards using the force $m'\dot{u}$ to run over water. The acceleration force is conservative. Second: a dissipative drag force $\sim c_{DB} b^2 (u-\bar{v})|\ u-\bar{v}|\varrho/2$. Third: a conservative lift force $\sim c_{LB} b^2 (u-\bar{v})|u-\bar{v}|\varrho/2$.

Eqs. (12) and (13) are plotted in Fig. 4. With increasing, shape-dependent Carnot's shock losses (see Fig. 3b), i.e. decreasing hydraulic efficiency (9), the efficiency drops below the limiting curve. For high dimensionless speeds, the propulsion efficiency of rowing is asymptotically identical to that of the fish. Only at low propulsion speeds the fish's movement is more efficient. The butterfly swimming stroke contains both propulsion types–anguilliform due to the body motion and rowing propulsion due to the arm stroke. The main difference between sweep rowing and sculling becomes clear by analyzing Eq. (12) and Fig. 4:

*As the outboard length l increases at same stroke rate and speed (*i.e. *sweep rowing* vs. *sculling), the Froude propulsion efficiency decreases.*

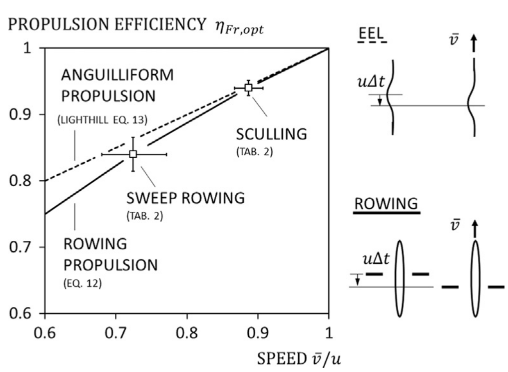

Fig. 4. Upper limit for the efficiency of any paddling or rowing motion in comparison with anguilliform propulsion.

At the same time the time-averaged thrust produced by all n times i blades, $=nil/\tau = niW_\infty m'/\tau$, has to overcome the shell drag D. As seen above, this is reached either by increasing the relative speed of the blade $W_\infty \approx u-\bar{v}$, or by increasing the blade size, i.e. the blade's added mass m'.

Hence, we now have, for the first time, a physical explanation for the observed (Kleshnev, 1999, 2011; Mang, 2008) difference in propulsion efficiencies between sweep rowing and sculling.

We now set about determining the constant Π in Eq. (6). Kleiber's law (4) and the definitions of mechanical and determined Froude propulsion efficiency result in $\overline{vT} = \varepsilon\, \eta_M\, \eta_{Fr} n m^{3/4}$. Using $\overline{T} = -\overline{D}$ and Eqs. (2) and (3), we gain the same result (6) independently and call this the
biomechanical equation for the speed of rowing

$$\bar{v} = \left(\frac{2}{\kappa \bar{c}_D}\right)^{1/3} \left(\frac{2\lambda}{1+\lambda}\right)^{1/3} (\eta\varepsilon)^{1/3}\, m^{1/36} n^{1/9}\, \varrho^{-1/9}. \qquad (14)$$

The total relative dissipation is given by $\eta := \eta_M \eta_H$. The quality of the hull and the propulsion system is hence given by the factor $(2\eta_M\eta_H/\kappa\bar{c}_D)^{1/3}$. Since $\lambda := \bar{v}/(\Omega l)$, Eq. (14) is an implicit, cubic equation for $\bar{v}$. However, since it has in most cases only one real root, it can be easily solved by a fixed-point iteration.

In conclusion: Provided Kleiber's law is valid for competitive rowers–which we indeed validate in Section 3 Eq. (14) is exact and captures all major influences on the speed of rowing.

3. Validation of allometric scaling of rowing

For the speed of rowing, the product $\eta\varepsilon$ in Eq. (14) has not yet been determined. However, it is possible to analyze actual average boat speeds by means of Eq. (14) and, in so doing, "measure" $\eta\varepsilon$. Our experimental set up is simple and at the same time of high quality: We employ the average boat speeds of the winning crews at the major events such Olympic, World Rowing Championship and Luzern International regattas in the decade from 1997 to 2007. At those major events all crews are trained as good as possible using the best available equipment at that time. Hence we in fact measure $\eta\varepsilon$ by employing Eq. (14) in the form

$$\frac{\bar{v}}{\bar{v}'} = \left(\frac{\eta\varepsilon}{\eta\varepsilon'}\right)^{1/3} \left(\frac{\eta_{Fr,opt}}{\eta'_{Fr,opt}}\right)^{1/3} \left(\frac{n}{n'}\right)^{1/9} \left(\frac{m}{m'}\right)^{1/36}. \qquad (15)$$

From the mentioned race times, we know the speed ratio $\bar{v}/\bar{v}'$ on the left hand side of Eq. (15), where the reference class is marked by a prime. Restricting to sculling or sweep rowing, in both cases $\eta_{Fr,opt}/\eta'_{Fr,opt} \approx 1$ and hence $\eta\varepsilon$ can be determined together with

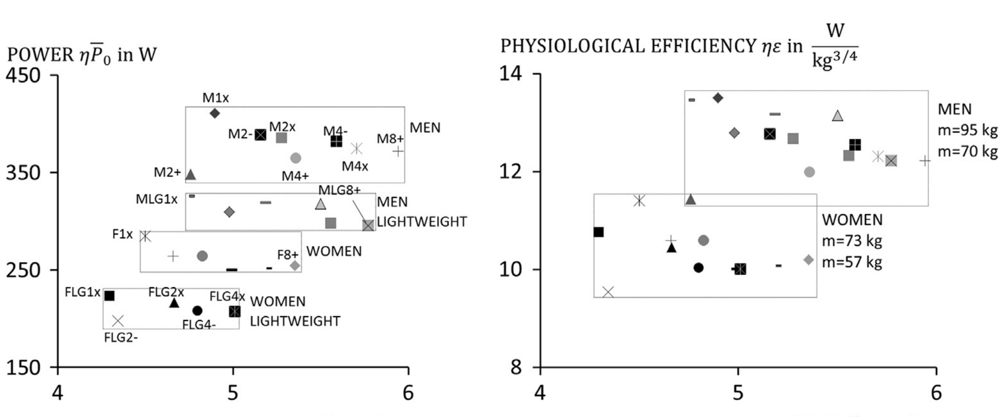

Fig. 5. Transmitted power per rower versus boat speed (left) and the product of physiological efficiency and efficiency vs. boat speed (right), as determined using Eq. (14). Data comes from the winning crews in the Olympic, World Rowing Championship and Luzern International regattas in the decade from 1997 to 2007. The left diagram shows the validity of allometric scaling for competitive rowers.

Eq. (14). In a similar way the propulsion efficiency η_{Fr} is "measurable" by analyzing race times.

If Kleiber's law (4) holds also for a competitive rower, there should be two values of the constant $\eta\varepsilon$, one for male and one for female rowers.

Fig. 5 left shows the results of our analysis, i.e., input power versus boat speed for the rowing classes indicated. Applying Kleiber's law (i.e. allometric scaling) to the data shows that lightweight and heavyweight crews have the same physiological behavior but differ according to sex (see Fig. 5 right). We can derive two constants for the decade from 1997 to 2007, one for female and one for male rowers. Despite the expected scatter of the data, as seen in Fig. 5, it is possible to determine an average value for calibration of the product of efficiency $\eta = \eta_M \eta_H$ and physiological efficiency for both women $\eta\varepsilon = 10.34\ W/\mathrm{kg}^{3/4}$, with a relative standard deviation of 4.62%, and men $\eta\varepsilon = 12.61\ W/\mathrm{kg}^{3/4}$, with a relative standard deviation of 4.45%. Hence, the race data confirms allometric scaling for rowers, which, in turn, confirms the validity of the speed of rowing equation.

In the same way we "measure" the Froude propulsion efficiency by employing Eqs. (14) and (15) to the mentioned data. The gained data, tabulated in Table 2, show the expected (see. Fig. 4) difference between sculling and sweep rowing. The values in Table 2 are gained from the same race results, were as Fig. 4 shows the analytical values of the Froude efficiency versus advance ratio. An average efficiency of 0.94, "measured" in the prescribed way for sculling, is gained for an advance ratio of 0.89 and an average efficiency of 0.84, "measured" for sweep rowing, needs an advance ratio of 0.72 as indicated by the axiomatic result Eq. (12). Indeed these advance ratios are typical for sculling and sweep rowing respectively (Kleshnev, 2001).

Table 2
Froude propulsion efficiency for sculling and sweep rowing. The original data are taken from the winning crews in the Olympic, World Rowing Championship and Luzern International regattas for the time period from 1991 to 2007.

	Froude propulsion efficiency η_{Fr}	
	Sculling	Sweep rowing
Men	0.95	0.88
Men lightweight	0.94	0.85
Women	0.93	0.83
Women lightweight	0.92	0.81
Mean	0.94	0.84
Rel. standard deviation	1.20%	3.07%

4. Application and conclusion—predicting and analyzing the speed of rowing

There are several possibilities for using our results, one of them we now consider in more detail. We use the speed of rowing equation to scale the race times of arbitrary rowing classes. Therefor the calibrated values of $\eta\varepsilon$ for the decade from 1997 to 2007 and the calibrated values for η_{Fr} for the time periode from 1991 to 2007 will be extrapolated to the considered time period from 1958 to 2007. The men's heavyweight single sculls were taken as reference, assuming similar shell geometry $1/\kappa\bar{c}_D$.

Fig. 6 shows the time-averaged rowing speed of winning crews versus year of the regatta (Olympic, World Championship and Luzern). We use the men's single sculls as reference and do an up-scaling for the men's eight and men's lightweight double sculls (Fig. 6a). Fig. 6b shows the down-scaling for the women's single sculls by means of the changed physiological efficiency and the up-scaling for the men's lightweight fours. Despite the scatter, the agreement of the predictions with actual race results is good, thus validating the postulated scale invariance of rowing.

In conclusion:

- A biomechanical equation for the speed of rowing was derived by two independent axiomatic methods, first a dimensional analysis and second a time averaged energy balance.
- By doing a hydrodynamic analysis on the basis of the momentum and energy balance, it could be shown that the Froude propulsion efficiency increases with increasing advance ratio.

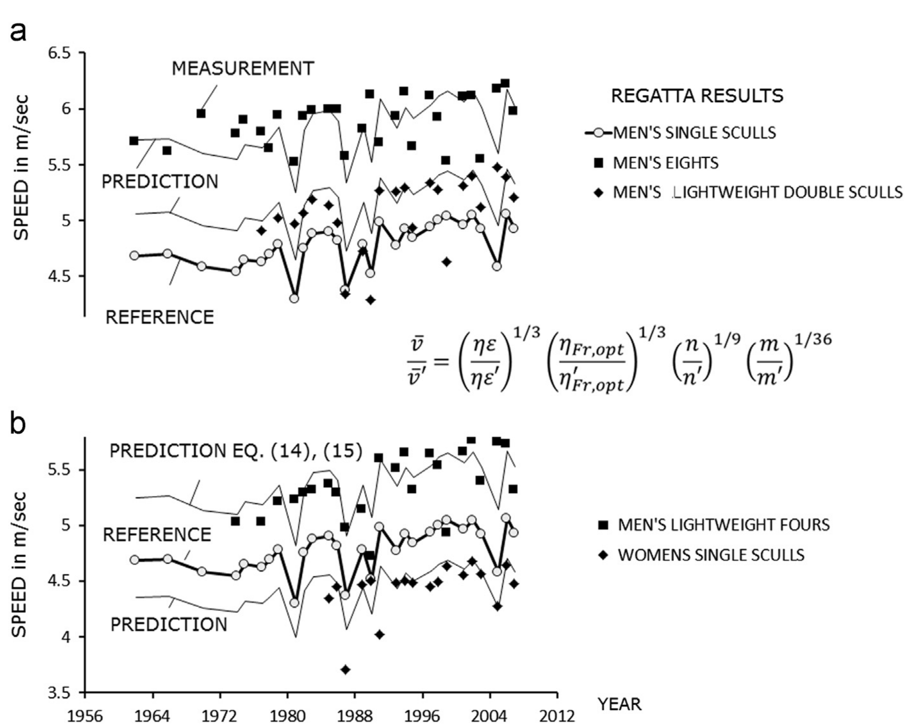

Fig. 6. Validation of the scaled speed of rowing (Eq. (14)) using measured winning race speeds. Reference is the men's single sculls result (thick solid line).

Hence, previous empirical results by other research could be explained in an axiomatic manner.

- By employing this equation and analyzing winning data of major regatta in the decade from 1997 to 2007, it is seen, that Kleiber's law is valid for both, male and female rowers.
- The physiological efficiency of any rower could be determined by analyzing the race times of the mentioned races.
- The results and in particular Eqs. (14) and (15) can be beneficial used to
  - Predict race times as it is shown in Fig. 6, serving as a validation.
  - Improve rowing equipment like hulls and oars employing Fig. 2 and Eq. (14).
  - See trends in boat speed over the years by plotting the dimensionless constant Π in Eq. (6) vs. time.
  - Apply the here developed analytical methods to other problems for biomechanical propulsion system.

Conflict of interest statement

Hereby we confirm that there is no conflict of interests arising, by submitting this paper to the Journal of Biomechanics. We disclose any financial and personal relationships with other people or organizations that could inappropriately influence (bias) our work.

Acknowledgment

The authors like to thank the German Research Foundation DFG for funding this research within the Collaborative Research Center (SFB) 805 “Control of Uncertainties in Load-Carrying Structures in Mechanical Engineering”.

Appendix A. Supporting information

Supplementary data associated with this article can be found in the online version at http://dx.doi.org/10.1016/j.jbiomech.2014.06.037.

References

Affeld, K., et al., 1993. Assessment of rowing efficiency. Int. J. Sports Med. 14 (Suppl 1), S39–S41.
Banavar, J.R., et al., 1999. Size and form in efficient transportation networks. Nature 399, 130–132.
Barenblatt, G.I., 2003. Scaling. Cambridge University Press, Cambridge.
Betz, A., 1959. Einführung in die Theorie der Strömungsmaschinen. Verlag G. Braun, Karlsruhe.
Bridgman, P.W., 1922. Dimensional Analysis, CT. Yale University Press, New Haven.
Cabrera, et al., 2006. A simple 1+ dimensional model of rowing mimics observed forces and motions. Hum Move Sci 25, 192–220.
Kleiber, M., 1932. Body Size and Metabolism. Hilgardia 6, 315–353.
Kleiber, M., 1975. The Fire of Life: An Introduction to Animal Energetics. Krieger Pub Co, Revised edition, New York.
Kleshnev, V., 1999. Propulsive efficiency of rowing. In: Proceedings of XVII International Symposium on Biomechanics in Sports, Perth, Australia.
Kleshnev, V., 2011. Biomechanics of rowing. In: Nolte, V. (Ed.), Rowing Faster, Second Ed. Human Kinetics, pp. 105–121.
Kleshnev, V., 2001. Stroke rate vs. distance in rowing during the Sydney Olympics, ⟨http://www.biorow.com/Papers_files/2000RaceRate.pdf⟩ (19/01/2008), Australian Rowing, Carine, 24(2), 18-22.
Lighthill, M.J., 1960. Note on the swimming of slender fish. J. Fluid Mech. 9, 305–317.
Mang, J., 2008. Ähnlichkeitstheorie beim Wettkampfrudern, Diplomarbeit Institut für Fluidsystemtechnik, Technische Universität Darmstadt.
McMahon, T.A., 1971. Rowing: a similarity analysis. Science 173, 349.
Newmann, J.N., 1977. Marine Hydrodynamics. The MIT Press, Cambridge, MA.
Prandtl, L., 1944. Strömungslehre, Zweite Auflage. Friedr. Vieweg & Sohn, Braunschweig.
Purcell, E.M., 1997. The efficiency of propulsion by a rotating flagellum. Proc. Natl. Acad. Sci. USA 94, 11307–11311.
Rayleigh, J., 1877. Theory of Sound. 1–2. Macmillan and Co, London.
Spurk, J.H., et al., 2008. Fluid Mechanics. Springer, Berlin, Heidelberg.
Taylor, G.I., 1953. Formation of a vortex ring by giving an impulse to a circular disk and then dissolving it away. J. Appl. Phys. 24, 104.
West, G.B., et al., 1999. The fourth dimension of life: fractal geometry and allometric scaling of organisms. Science 284, 1677–1679.